Section 5.4:
Rank of a Matrix

Ideas in this section…

Given an $m \times n$ matrix A…

- Definition of row space and column space of A
- Definition of null space and image space of A
- Definition of the eigenspace of A
- col(A) = im(A)
- dim[row(A)] = dim[col(A)] = rank(A)
- Rank-Nullity Theorem: rank(A) + dim[null(A)] = n
- How can you find basis for these spaces?
- How can you tell if a vector is in these spaces?

Special Subspaces of $\mathbb{R}^n$: Row Space / Column Space

Def: Let A be an $m \times n$ matrix.

1) The row space of A (denoted by $row\,A$) is the subspace of $\mathbb{R}^n$ spanned by the rows of A.

2) The column space of A (denoted by $col\,A$) is the subspace of $\mathbb{R}^m$ spanned by the columns of A.

Ex 1: If $A = \begin{bmatrix} 1 & 2 & 5 \\ -3 & 0 & 4 \end{bmatrix}$, then…

$$row(A) = span\{\,(1,2,5)\,,(-3,0,4)\,\} = \{c_1(1,2,5) + c_2(-3,0,4) \mid c_1, c_2 \in \mathbb{R}\,\}$$

$$col(A) = span\left\{\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \end{bmatrix}\right\} = \left\{c_1\begin{bmatrix} 1 \\ -3 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} 5 \\ 4 \end{bmatrix} \mid c_1, c_2, c_3 \in \mathbb{R}\right\}$$

Special Subspaces of $\mathbb{R}^n$: Row Space / Column Space

Lemma 5.4.1

Let A and B denote $m \times n$ matrices.

1. *If $A \to B$ by elementary row operations, then* $\text{row } A = \text{row } B$.
2. *If $A \to B$ by elementary column operations, then* $\text{col } A = \text{col } B$.

<u>Proof of 1:</u>

Special Subspaces of $\mathbb{R}^n$: Row Space / Column Space

Lemma 5.4.2

If R is a row-echelon matrix, then

1. *The nonzero rows of R are a basis of* $\text{row}\, R$.
2. *The columns of R containing leading ones are a basis of* $\text{col}\, R$.

Discuss:

Special Subspaces of $\mathbb{R}^n$: Row Space / Column Space

Lemma 5.4.1

Let A and B denote $m \times n$ matrices.

1. *If $A \to B$ by elementary row operations, then $\text{row } A = \text{row } B$.*
2. *If $A \to B$ by elementary column operations, then $\text{col } A = \text{col } B$.*

Lemma 5.4.2

If R is a row-echelon matrix, then

1. *The nonzero rows of R are a basis of $\text{row } R$.*
2. *The columns of R containing leading ones are a basis of $\text{col } R$.*

Recall: Def: Given an $m \times n$ matrix A. Row reduce A to an echelon form matrix R. Then the rank of A is the number of leading 1's in matrix R.

With Lemma 5.4.2 we can fill a gap in the definition of the rank of a matrix given in Chapter 1. Let A be any matrix and suppose A is carried to some row-echelon matrix R by row operations. Note that R is not unique. In Section 1.2 we defined the **rank** of A, denoted $\text{rank } A$, to be the number of leading 1s in R, that is the number of nonzero rows of R. The fact that this number does not depend on the choice of R was not proved in Section 1.2. However part 1 of Lemma 5.4.2 shows that

$$\text{rank } A = \dim(\text{row } A)$$

and hence that $\text{rank } A$ is independent of R.

Special Subspaces of $\mathbb{R}^n$: Row Space / Column Space

Theorem 5.4.1: Rank Theorem

Let A denote any $m \times n$ matrix of rank r. Then

$$\dim(\text{col } A) = \dim(\text{row } A) = r$$

Moreover, if A is carried to a row-echelon matrix R by row operations, then

1. *The r nonzero rows of R are a basis of* row A.
2. *If the leading* 1*s lie in columns $j_1, j_2, \dots, j_r$ of R, then columns $j_1, j_2, \dots, j_r$ of A are a basis of* col A.

Discuss:

Special Subspaces of $\mathbb{R}^n$: Row Space / Column Space

<u>Ex 2</u>: Compute the rank of $A = \begin{bmatrix} 1 & 2 & 2 & -1 \\ 3 & 6 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{bmatrix}$ and find basis for $row(A)$ and $col(A)$.

Reducing a Set to a Basis

Ex 3 (Ex 12 from last lecture): Find a basis and calculate the dimension of

$$span\{\ (-1,2,1,0), (2,0,3,-1), (4,4,11,-3), (3,-2,2,-1)\ \}$$

Reducing a Set to a Basis

Ex 4 (Ex 13 from last lecture): Find a basis of P_3 in the spanning set

$$\{1,\ x+x^2,\ 2x-3x^2,\ 1+3x-2x^2,\ x^3\}$$

Special Subspaces of $\mathbb{R}^n$: Row Space / Column Space

Corollary 5.4.1

If A is any matrix, then $\operatorname{rank} A = \operatorname{rank}(A^T)$.

Corollary 5.4.2

If A is an $m \times n$ matrix, then $\operatorname{rank} A \leq m$ *and* $\operatorname{rank} A \leq n$.

Why?

Special Subspaces of $\mathbb{R}^n$: Null Space of a Matrix

Def: Given an $m \times n$ matrix A, the null space of A is

$$null(A) = \{ \vec{x} \in \mathbb{R}^n \mid \mathrm{A}\vec{x} = \vec{0} \}$$

Special Subspaces of $\mathbb{R}^n$: Null Space of a Matrix

Ex 5: If $A = \begin{bmatrix} 1 & 3 & -1 \\ 7 & 2 & 2 \\ 5 & -4 & 4 \\ 11 & -5 & 7 \end{bmatrix}$,

a) show that $\begin{bmatrix} -8 \\ 9 \\ 19 \end{bmatrix} \in null(A)$

Special Subspaces of $\mathbb{R}^n$: Null Space of a Matrix

Ex 5: If $A = \begin{bmatrix} 1 & 3 & -1 \\ 7 & 2 & 2 \\ 5 & -4 & 4 \\ 11 & -5 & 7 \end{bmatrix}$,

b) show that $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} \notin null(A)$

Special Subspaces of $\mathbb{R}^n$: Null Space of a Matrix

Def: Given an $m \times n$ matrix A, the null space of A is

$$null(A) = \{\, \vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\,\}$$

Result: If A is an $m \times n$ matrix A, $null(A)$ is a subspace of $\mathbb{R}^n$.

Proof:

Special Subspaces of $\mathbb{R}^n$: Null Space of a Matrix

Ex 6: If $A = \begin{bmatrix} 1 & 3 & -1 \\ 7 & 2 & 2 \\ 5 & -4 & 4 \\ 11 & -5 & 7 \end{bmatrix}$, find $null(A)$.

Special Subspaces of $\mathbb{R}^m$: Image Space of A

Def: Given an $m \times n$ matrix A, the image space of A is

$$im(A) = \{ A\vec{x} \mid \vec{x} \in \mathbb{R}^n \}$$

Special Subspaces of $\mathbb{R}^m$: Image Space of A

Ex 7: If $A = \begin{bmatrix} 1 & 3 & -1 \\ 7 & 2 & 2 \\ 5 & -4 & 4 \\ 11 & -5 & 7 \end{bmatrix}$, find some vectors in $im(A)$.

Columns, columns span

Special Subspaces of $\mathbb{R}^m$: Image Space of A

Result: If A is an $m \times n$ matrix, $im(A)$ is a subspace of $\mathbb{R}^n$.

Proof (2 ways):

Special Subspaces: Null Space and Image Space of A

Ex 8: If $A = \begin{bmatrix} 1 & -2 & 1 & 1 \\ -1 & 2 & 0 & 1 \\ 2 & -4 & 1 & 0 \end{bmatrix}$, find basis for $null(A)$ and $im(A)$, and find their dimensions.

Special Subspaces of $\mathbb{R}^n$: Eigenspace of a Matrix

Recall:

Def: Let A be an $n \times n$ matrix. If there is a number λ and a non-zero $n \times 1$ column matrix $\boldsymbol{x}$ such that

$$A\boldsymbol{x} = \lambda\boldsymbol{x}$$

Then $\boldsymbol{x}$ is called an eigenvector of A and λ is called an eigenvalue of A.

Special Subspaces of $\mathbb{R}^n$: Eigenspace of a Matrix

<u>Def</u>:

If A be an $n \times n$ matrix and λ is an eigenvalue of A,
the set of all eigenvectors of A corresponding to the eigenvalue λ together with the zero vector $\vec{0}$ is called the <u>eigenspace of A corresponding to the eigenvalue λ</u>.

Notes:

- Notation: $E_\lambda(A)$
- $E_\lambda(A) = null(\lambda I - A)$

Special Subspaces of $\mathbb{R}^n$: Eigenspace of a Matrix

Result:

If A be an $n \times n$ matrix and λ is an eigenvalue of A.

Then $E_\lambda(A) = \{\, \vec{x} \in \mathbb{R}^n \mid A\vec{x} = \lambda\vec{x} \,\}$ is a subspace of $\mathbb{R}^n$.

Proof (in 2 ways):

Special Subspaces of $\mathbb{R}^n$: Eigenspace of a Matrix

Ex 9: For the matrix $A = \begin{bmatrix} 11 & 0 & 21 \\ 0 & -1 & 0 \\ -8 & 0 & -15 \end{bmatrix}$,…

$$E_{\lambda=-1}(A) = \left\{ s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -7/4 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; s,t \in \mathbb{R} \right\}$$

$$E_{\lambda=-3}(A) = \left\{ r\begin{bmatrix} -3/2 \\ 0 \\ 1 \end{bmatrix} \;\middle|\; r \in \mathbb{R} \right\}$$

The Rank-Nullity Theorem

Theorem 5.4.2

Let A denote an $m \times n$ matrix of rank r. Then

1. *The $n - r$ basic solutions to the system $A\mathbf{x} = \mathbf{0}$ provided by the gaussian algorithm are a basis of* $\operatorname{null}(A)$, *so* $\dim[\operatorname{null}(A)] = n - r$.
2. *Theorem 5.4.1 provides a basis of* $\operatorname{im}(A) = \operatorname{col}(A)$, *and* $\dim[\operatorname{im}(A)] = r$.

3. (Rank-Nullity Theorem): rank(A) + nullity(A) = n

Discuss:

More Examples from the Online Homework

Ex 9a: Consider the matrix $A = \begin{bmatrix} 3 & 6 & -6 \\ 1 & 4 & -4 \\ 1 & 0 & 5 \\ 1 & 1 & -3 \end{bmatrix}$.

Which of the following vectors $\vec{b}$ are in col A ?

If $\vec{b} \in col\ A$, find a vector $\vec{x}$ such that $A\vec{x} = \vec{b}$.

a) $\mathbf{b} = \begin{bmatrix} 3 \\ 1 \\ -9 \\ 5 \end{bmatrix}$

More Examples from the Online Homework

Ex 9b: Consider the matrix $A = \begin{bmatrix} 3 & 6 & -6 \\ 1 & 4 & -4 \\ 1 & 0 & 5 \\ 1 & 1 & -3 \end{bmatrix}$.

Which of the following vectors $\vec{b}$ are in col A ?

If $\vec{b} \in col\ A$, find a vector $\vec{x}$ such that $A\vec{x} = \vec{b}$.

b) $\mathbf{b} = \begin{bmatrix} 10 \\ -1 \\ 1 \\ 3 \end{bmatrix}$

More Examples from the Online Homework

Ex 10:

Consider the following matrix A and its reduced row-echelon form:

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 & 1 & -5 \\ 3 & 3 & 6 & 6 & 3 & -15 \\ 2 & 2 & 4 & 4 & 2 & -10 \\ 3 & 6 & 15 & 12 & 9 & -15 \end{bmatrix} \quad \text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & 0 & -1 & -5 \\ 0 & 1 & 3 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find the dimensions of $\text{row}(A)$, $\text{null}(A)$, and $\text{im}(A)$, and give a basis for each of them.

Then verify the rank-nullity theorem.

What you need to know from the book

Book reading

Section 5.1: pages 264-269 all

Section 5.4: pages 290-296 all

Problems you need to know how to do from the book

Section 5.1 #'s 12-14

Section 5.2 #'s 3, 17, 18

Section 5.4 #'s 1-15

Section 6.4 # 2